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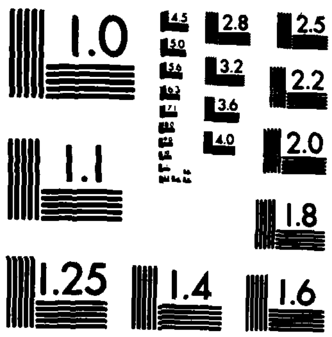

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FOR

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For the period 4/1/81 - 3/31/82

MIXED FINITE ELEMENT METHODS
WITH APPLICATIONS TO FLOW
AND OTHER PROBLEMS

Prepared for the Air Force Office of Scientific Research

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This is a report on progress of work supported by the grant during the period 1 Apr 81-31 Mar 82. Three problems were considered. These were finite element approximations to the inhomogeneous Navier-Stokes equations, for potential flows, and for acoustic eigenvalue problems. In all cases both theoretical error estimates and computer codes implementing the best algorithm were developed. Other activities sponsored by the grant, i.e., student research and conference talks, are also reported on.		

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Table of Contents

I. ABSTRACT

We report on progress of work supported by the Air Force Office of Scientific Research Grant No. AFOSR-80-0083 during the period 4/1/81 - 3/31/82. Three problems were considered. These were finite element approximations to the inhomogeneous Navier-Stokes equations, for potential flows, and for acoustic eigenvalue problems. In all cases both theoretical error estimates and computer codes implementing the best algorithm were developed. We also report on other activities sponsored by the grant, i.e. student research and conference talks.

II. PROGRESS REPORT

1. INHOMOGENEOUS NAVIER STOKES EQUATIONS

Finite element methods for stationary viscous incompressible flows were considered. Specifically, we consider the problem of finding a velocity field $\underline{u}$ and a presure field p which satisfy

$$\nabla \cdot \underline{u} = g \quad \text{in} \quad \Omega$$

$$\nu\Delta\underline{u} - \underline{u}\cdot\nabla\underline{u} + \nabla p = \underline{f} \quad \text{in} \quad \Omega \tag{1}$$

$$\underline{u} \cdot \underline{n} = q \quad \text{on} \quad \Gamma$$

where Ω is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$ with boundary Γ, and where g, $\underline{f}$, and q are given functions. Previous work [1] on the topic consider the case $g = 0$ and $q = 0$ only. The case $q \neq 0$ is important since usually such flows are driven at the boundaries, e.g. by inflows. Our work on this topic included the following accomplishments:

a) Under natural hypothesis on $\underline{f}$, g, and q, the existence and uniqueness of weak solution of (1) was proven. Adding stability hypotheses on the finite element spaces, the existence and uniqueness of approximate finite element solutions was also displayed.

b) Optimal error estimates for the finite element discretization were derived.

c) The convergence behavior of iterative methods, i.e. the Newton,chord and simple iteration methods for the solution of the discrete nonlinear algebraic equations resulting from the finite element discretization were analyzed. In particular, it was shown that the Newton method was quadratically convergent when one is close enough to the exact solution of the discrete equations, and it was also shown that a simple iteration scheme is globaly convergent whenever the solution of (1) is unique.

d) Computer programs were written implementing some particular choices of finite element spaces. These programs verified the theoretical results and also served to illustrate the implementation of the algorithms developed.

These results were reported in Paper 4 of Section III below.

A crucial step in developing good algorithms for the approximation of the solution of (1) is choosing finite element spaces which satisfy the stability hypothesis alluded to in (a) above. We choose spaces $\underline{V}^h$ and S^h in which to seek an approximation $\underline{u}^h$ and p^h, respectively, to (1). Then the stability hypothesis takes the form

$$\sup_{\underline{v}^h \in \underline{V}^h} \frac{\int_\Omega \psi^h \operatorname{div} \underline{v}^h d\Omega}{\|\underline{v}^h\|_1} \geq \gamma^h \|\psi^h\|_0 \quad \forall \psi^h \in S^h.$$

where $\|\cdot\|_1$ denotes the H^1-Sobolev norm. The crucial question is whether or not

$$\gamma^h \geq \gamma_0 > 0$$

uniformly in h, i.e. γ_0 is independent of h. For some obvious choices of $\underline{V}^h$ and S^h, $\gamma_0 = 0$ and these are discarded. For some other choices, such as bilinear velocities and constant pressures, $\gamma_0 \neq 0$ once the "checkerboard" pressure mode is removed. However, there is some evidence that γ^h depends on h, i.e. $\gamma^h \to 0$. This results in possibly bad pressure approximations which must be filtered in order to obtain useful pressures. On the other hand, the velocity approximations are optimally accurate without any filtering. Computer programs implementing a variety of low order elements have been developed. These programs produce accurate velocity approximations. This work is reported on in Papers 1, 2, and 6 of Section III below.

2. MIXED FINITE ELEMENT METHODS FOR POTENTIAL FLOWS

This work involves the approximation of problems of the type

$$\begin{aligned} \nabla \cdot \underline{u} &= 0 \quad \text{in} \quad \Omega \\ \underline{u} &= \nabla\phi \quad \text{in} \quad \Omega \\ \underline{u} \cdot \underline{n} &= f \quad \text{on} \quad \Gamma_N \\ \phi &= g \quad \text{on} \quad \Gamma_D \end{aligned} \tag{2}$$

where $\Gamma_N \cap \Gamma_D = \Gamma$, the boundary of Ω, f and g are given functions, and $\underline{u}$ and ϕ are, for instance, an unknown velocity field and potential field. The simple problem (2) is equivalent to

$$\begin{aligned} \Delta\phi &= 0 \quad \text{in} \quad \Omega \\ \phi &= g \quad \text{on} \quad \Gamma_N \\ \frac{\partial\phi}{\partial n} &= f \quad \text{on} \quad \Gamma_0. \end{aligned} \tag{3}$$

However, we are intersted in discretizing (1) directly, since in more general settings, problems of the type (2) may not always be recast into a form similar to (3). Previous work [2] on finite element methods for the approximation to the solution of (2) resulted in error estimates for the error $\underline{u} - \underline{u}^h$, $\underline{u}^h$ being the discrete solution, in the norm

$$\|\underline{v}\|_* = \|\underline{v}\|_0 + \|\mathrm{div}\ \underline{v}\|_0$$

where $\|\cdot\|_0$ denotes the L^2-norm. No error estimates were obtainable for $\|\underline{u}-\underline{u}^h\|_0$, which is physically of much greater interest than estimates for $\|\underline{u}-\underline{u}^h\|_*$. The thrust of our work was to examine conditions under which optimally accurate approximations in the norm $\|\cdot\|_0$ are obtainable for the problem (2). The highlights of this work are the following:

(a) Optimally accurate approximations are obtainable whenever the subspace $\underline{V}^h$ and S^h in which we seek our approximate solution $\underline{u}^h$ and ϕ^h, respectively, satisfy the inclusion property $S^h = \mathrm{div}(\underline{V}^h)$ and the decomposition property: every $\underline{u}^h \in \underline{V}^h$ may be written in the form

$$\underline{v}^h = \underline{w}^h + \underline{z}^h$$

where

$$\nabla \cdot \underline{z}^h = 0$$

and

$$\beta^h\|\underline{w}^h\|_0 \leq \|\mathrm{div}\ \underline{w}^h\|_{-1}$$

where

$$\|f\|_{-1} = \sup_{\substack{\psi \in H_0^1(\Omega) \\ \psi \neq 0}} \frac{\int_\Omega f\psi d\Omega}{\|\psi\|_1}.$$

Again, bounding β^h by $\beta^h > \beta_0$ uniformly in h is crucial to obtaining optimally accuate approximations.

(b) An example of a pair $\underline{V}^h$ and S^h satisfying the above properties was given.

(c) Computation using the spaces of (b) were performed, verifying the optimal theoretical results. Computations using finite element spaces which violate the conditions of (a) were also performed, and non-optimal or divergent approximations resulted. This work was reproted on in Paper 2 of Section III.

3. EIGENVALUE APPROXIMATIONS

Work was completed on the study of finite element approximations for acoustic eigenvalue problems. Here we consider problems of the type

$$\begin{aligned} \Delta\phi + \omega\phi &= 0 \quad \text{in} \quad \Omega \\ \phi &= 0 \quad \text{on} \quad \Gamma \end{aligned}$$

recast into the weak form

$$\int_\Omega \underline{u}\cdot\underline{v}d\Omega - \int_\Omega \phi \operatorname{div}\underline{v}d\Omega = 0 \quad \forall\underline{v} \in \underline{V}$$

$$-\int_\Omega \psi \operatorname{div}\,\underline{u}d\Omega = \omega\int_\Omega \phi\psi d\Omega \quad \forall\psi \in S \tag{4}$$

for suitable Hilbert spaces $\underline{V}$ and S. Error estimates for finite element approximations of (4) were performed. Under the conditions discussed in subsection 2 above, these estimates were optimal. Computer codes implementing stable algorithms were developed, everywhere verifying the theoretical results. This work was reported in Paper 1 of Section III.

4. REFERENCES

[1] Girault, V. and P. Raviart, Finite Element Approximations of the Navier-Stokes Equation, 1979, Springer.

[2] Raviart, P. and J. Thomas, "A mixed finite element method for 2-nd order elliptic problems", Lecture Notes in Mathematics, No. 606, 1977, Springer.

III. ACTIVITIES

PAPERS PREPARED UNDER GRANT SPONSORSHIP (Copies of these papers have been forwarded to AFOSR)

1. "Mixed finite element methods with applications to acoustic and flow problems"; Proc. 5th AIAA Comp. Fluids Conf., AIAA CP814, pp. 265-271; [by G. Fix, M. Gunzburger, R. Nicolaides, J. Peterson].
2. "On mixed finite element methods for first order elliptic systems", Num. Math., 37, 1981, pp. 29-48; [by G. Fix, M. Gunzburger, R. Nicolaides].
3. "On conforming finite element methods for incompressible viscous flow problems"; Comp. & Math. with Appls., 8, 1982, pp. 167-179; [by M. Gunzburger, R. Nicolaides, J. Peterson].
4. "On conforming finite element methods for the inhomogeneous stationary Navier-Stokes equations", to appear; [by M. Gunzburger, J. Peterson].
5. "An application of mixed finite element methods to the stability of the incompressible Navier-Stokes equation; [by J. Peterson].
6. "New results in the finite element solutions of steady viscous flows"; The Mathematics of Finite Elements and Applications IV, Academic Press, 1982, pp. 463-470; [by M. Gunzburger and R. Nicolaides].

Note: 1, 2, 3, and 6 were reported on in the previous interim report, and are noted here to note that they have appeared in the open literature. 4 and 5 were prepared during the period reported in this report.

STUDENTS PARTIALLY SPONSORED BY GRANT

Jerome Eastham (Ph.D., 1981, Tennessee) Thesis title: "On the finite element method in anisotropic Sobolev spaces".

Georges Guirguis (Ph.D., Tennessee, expected in 1983) Thesis topic: "Finite element approximations to the Stokes equations in exterior domains".

TALKS PRESENTED UNDER GRANT SPONSORSHIP

1. AMS Meeting, May 15-16, 1981, Pittsburgh, Pa., "Finite element approximations of the Navier-Stokes equations" [M. Gunzburger].
2. AIAA Computational Fluids Conference, June 22-23, 1981, Palo Alto, Ca., "On mixed finite element methods for acoustic and flow problems" [M. Gunzburger].

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